

DUAL MEAN VALUE PROBLEM FOR COMPLEX POLYNOMIALS

VLADIMIR DUBININ AND TOSHIYUKI SUGAWA

ABSTRACT. We consider an extremal problem for polynomials, which is dual to the well-known Smale mean value problem. We give a rough estimate depending only on the degree.

1. INTRODUCTION

Let P be a complex polynomial of degree $d \geq 2$, that is, P is a polynomial map on the complex plane \mathbb{C} of the form

$$P(z) = c_d z^d + c_{d-1} z^{d-1} + \cdots + c_1 z + c_0$$

for complex coefficients c_0, \dots, c_d with $c_d \neq 0$. We denote by \mathcal{P}_d the set of complex polynomials of degree d . We say that $\zeta \in \mathbb{C}$ is a *critical point* of P if $P'(\zeta) = 0$. The image $P(\zeta)$ of a critical point ζ under P is called a *critical value*. We denote by $\text{Crit}(P)$ the set of critical points of P . In 1981, Smale [10] proved the following inequality in connection with root-finding algorithms.

Theorem A (Smale). *Let P be a polynomial of degree $d \geq 2$ over \mathbb{C} and suppose that $z \in \mathbb{C}$ is not a critical point of P . Then there exists a critical point ζ of P such that*

$$(1.1) \quad \left| \frac{P(\zeta) - P(z)}{\zeta - z} \right| \leq 4|P'(z)|.$$

In the same paper, Smale asked whether the factor 4 can be replaced by 1 or even by $1 - 1/d$. See also [9] and [7, § 7.2] for background and further references.

To simplify expressions and to emphasize invariance, we introduce some notation. For $\zeta \in \text{Crit}(P)$ and $z \in \mathbb{C} - \text{Crit}(P)$, we define $Q(P, z, \zeta)$ by

$$Q(P, z, \zeta) = \frac{P(z) - P(\zeta)}{(z - \zeta)P'(z)}.$$

It is easy to verify the invariance relation $Q(\tilde{P}, \tilde{z}, \tilde{\zeta}) = Q(P, z, \zeta)$ for $z = a\tilde{z} + b$, $\zeta = a\tilde{\zeta} + b$ and $\tilde{P}(\tilde{z}) = AP(a\tilde{z} + b) + B$ for constants a, b, A, B with $aA \neq 0$.

We further set

$$S(P, z) = \min\{|Q(P, z, \zeta)| : \zeta \in \text{Crit}(P)\}$$

Date: June 22, 2009, *File:* dubinin-sugawa0906.tex.

2000 Mathematics Subject Classification. Primary 30C10; Secondary 30C55.

Key words and phrases. Smale's mean value conjecture, critical point.

The present research was partially supported by the Russian Foundation for Basic Research (grant no. 08-01-00028), the Far-Eastern Branch of RAS (grant no. 09-I-P4-02), and the JSPS Grant-in-Aid for Scientific Research (B), 17340039.

for $z \in \mathbb{C}$ and

$$K(d) = \sup\{S(P, z) : P \in \mathcal{P}_d, z \in \mathbb{C}\}.$$

Smale's theorem says that $K(d) \leq 4$, while the example $P_0(z) = z^d - dz$ shows that $K(d) \geq 1 - 1/d$. Smale's problem asks to find the value of $K(d)$ and Smale's conjecture can be stated as $K(d) = 1 - 1/d$. This conjecture has been confirmed for degrees $d = 2, 3, 4$ (Tischler [11]), for $d = 5$ (Crane [3]). Sendov and Marinov [8] claim that the conjecture is true for $d \leq 10$ by massive numerical computations, which one cannot check easily. Meanwhile, some improvements were made for Smale's theorem, see [1], [2], [5], [4]. Note that all the known estimates $K(d) \leq \tilde{K}(d)$ satisfy $\liminf_{d \rightarrow \infty} \tilde{K}(d) \geq 4$.

We may pose a dual problem to Smale's mean value problem: Consider the quantity

$$T(P, z) = \max\{|Q(P, z, \zeta)| : \zeta \in \text{Crit}(P)\}$$

for $z \in \mathbb{C}$ and find the value

$$L(d) = \inf\{T(P, z) : P \in \mathcal{P}_d, z \in \mathbb{C}\}.$$

For the polynomial $P^*(z) = (z + 1)^d - 1$, we have $T(P^*, 0) = 1/d$, and hence $L(d) \leq 1/d$.

We should now mention Tischler's strong form of Smale's conjecture:

$$(1.2) \quad \min \left\{ \left| Q(P, z, \zeta) - \frac{1}{2} \right| : \zeta \in \text{Crit}(P) \right\} \leq \frac{1}{2} - \frac{1}{d}, \quad P \in \mathcal{P}_d.$$

By the triangle inequality $||Q(P, z, \zeta)| - 1/2| \leq |Q(P, z, \zeta) - 1/2|$, (1.2) would imply the inequality:

$$\max \left\{ S(P, z) - \frac{1}{2}, \frac{1}{2} - T(P, z) \right\} \leq \frac{1}{2} - \frac{1}{d},$$

in other words, $S(P, z) \leq 1 - 1/d$ and $1/d \leq T(P, z)$. Since the inequalities $1 - 1/d \leq K(d)$ and $L(d) \leq 1/d$ trivially hold, the relations $K(d) = 1 - 1/d$ and $L(d) = 1/d$ would follow from (1.2).

It is known that (1.2) is valid for $d \leq 4$, see Tischler [11], where the case $d = 4$ is attributed to J.-C. Sikorav. In particular, we have $L(d) = 1/d$ for $d = 2, 3, 4$. Unfortunately, (1.2) does not hold in general. Indeed, Tyson [12] revealed that (1.2) does not hold for every $d \geq 5$.

In the present note, we give the rough estimate $L(d) \geq 1/(d4^d)$ as the first step towards the conjecture $L(d) = 1/d$.

Theorem 1. *Let P be a polynomial of degree $d \geq 2$ over \mathbb{C} and suppose that $z \in \mathbb{C}$ is not a critical point of P . Then there exists a critical point ζ of P such that*

$$\frac{|P'(z)|}{d4^d} \leq \left| \frac{P(\zeta) - P(z)}{\zeta - z} \right|.$$

We proved this theorem when the first author visited Hiroshima University, where the second author was working, in July 2007. We learnt from T. W. Ng that he considered the same problem independently and that he showed the inequality $L(d) > 0$ for each d in the same year by using the notion of amoebae (see [6]). Since there is no explicit lower bound for $L(d)$ in the literature so far, our result seems to be meaningful even though the bound is too far from the conjectured one.

Note also that the conjecture is true for the special case when P is conservative, that is, $P(\zeta) = \zeta$ for every $\zeta \in \text{Crit}(P)$. See the last corollary in [11, p. 455], where Tischler attributes its parent theorem to Yoccoz.

2. PROOF OF THE THEOREM

By the transformation $\tilde{P}(\tilde{z}) = [P(\tilde{z} + z_0) - P(z_0)]/c_d$, one can easily see that Theorem 1 is equivalent to the following assertion.

Theorem 2. *Let $\zeta_1, \dots, \zeta_{d-1}$ be the critical points of a monic polynomial P of degree $d \geq 2$ with $P'(0) \neq 0$. Then*

$$\max_{1 \leq j \leq d-1} \left| \frac{P(\zeta_j)}{\zeta_j P'(0)} \right| \geq \frac{1}{d4^d}.$$

Proof. Since $P'(z) = d(z - \zeta_1) \cdots (z - \zeta_{d-1})$, the relation

$$(2.1) \quad P'(0) = d(-1)^{d-1} \zeta_1 \cdots \zeta_{d-1}$$

holds.

For $t > 0$, we set $\Delta(t) = \{w \in \widehat{\mathbb{C}} : |w| > t\}$, where $\widehat{\mathbb{C}}$ stands for the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Let

$$R = \max_{1 \leq j \leq d-1} |P(\zeta_j)|^{1/d}$$

and set $A = \{z \in \mathbb{C} : |P(z)| \leq R^d\}$. Note that A is a continuum containing 0 and all the critical points $\zeta_1, \dots, \zeta_{d-1}$. Since $P : \mathbb{C} - A \rightarrow \Delta(R^d) - \{\infty\}$ is an unbranched covering map of degree d , one can take a single-valued analytic branch $F(w)$ of $P^{-1}(w^d)$ on $\Delta(R)$ with $F(w) = w + O(1)$ as $w \rightarrow \infty$. Note that the function F satisfies the relation

$$(2.2) \quad P(F(w)) = w^d, \quad |w| > R$$

and that $F(\Delta(R)) = \widehat{\mathbb{C}} - A$. We now show that F is univalent in $\Delta(R)$. Suppose, to the contrary, that $F(w_0) = F(w_1)$ for distinct points w_0 and w_1 in $\Delta(R)$. Obviously, $z_0 = F(w_0)$ is not the point at infinity. Since P has no critical point in $\mathbb{C} - A$, P is univalent in a small neighborhood V of z_0 . On the other hand, by (2.2), we have $w_0^d = w_1^d$, and thus, $w_1 = \tau w_0$ for a complex number $\tau \neq 1$ with $\tau^d = 1$. We consider the function $F_1(w) = F(\tau w)$ in $|w| > R$. Then $F(w_0) = F_1(w_0)$. Since $F_1(w) = \tau w + O(1)$ as $w \rightarrow \infty$, F_1 and F are not identically equal on $\Delta(R)$. Therefore, $F(w) \neq F_1(w)$ for $w \neq w_0$ close enough to w_0 . By (2.2), we have $P(F(w)) = P(F_1(w))$ for $w \in \Delta(R)$. This is impossible because P is univalent in V . Thus we have shown that F is univalent in $\Delta(R)$.

Let f be a function f on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by $f(z) = R/F(R/z)$. Since F has no zero in $\Delta(R)$, we see that f is analytic in \mathbb{D} and that $f(0) = f'(0) - 1 = 0$. From what we saw above, we also obtain that f is univalent in \mathbb{D} . The Koebe one-quarter theorem implies that the image $f(\mathbb{D})$ contains the disk $|w| < 1/4$. Since $\zeta_j \in A = \widehat{\mathbb{C}} - F(\Delta(R))$, we have

$$(2.3) \quad \frac{R}{|\zeta_j|} \geq \frac{1}{4}, \quad j = 1, 2, \dots, d-1.$$

We may assume that $R = |P(\zeta_1)|^{1/d}$. Then, by (2.1) and (2.3), we have

$$\max_{1 \leq j \leq d-1} \left| \frac{P(\zeta_j)}{\zeta_j P'(0)} \right| \geq \left| \frac{P(\zeta_1)}{\zeta_1 P'(0)} \right| = \frac{R^d}{|\zeta_1| \cdot d |\zeta_1 \cdots \zeta_{d-1}|} \geq \frac{1}{d4^d}.$$

The proof is now complete. \square

REFERENCES

1. A. F. Beardon, D. Minda, and T. W. Ng, *Smale's mean value conjecture and the hyperbolic metric*, Math. Ann. **322** (2002), 623–632.
2. A. Conte, E. Fujikawa, and N. Lakic, *Smale's mean value conjecture and the coefficients of univalent functions*, Proc. Amer. Math. Soc. **135** (2007), 3295–3300.
3. E. Crane, *A computational proof of the degree 5 case of Smale's mean value conjecture*, preprint.
4. ———, *A bound for Smale's mean value conjecture for complex polynomials*, Bull. London Math. Soc. **39** (2007), 781–791.
5. E. Fujikawa and T. Sugawa, *Geometric function theory and Smale's mean value conjecture*, Proc. Japan Acad. **82** (2006), 97–100.
6. T. W. Ng, *Smale's mean value conjecture and amoebae*, preprint (2007).
7. Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press Oxford University Press, Oxford, 2002.
8. Bl. Sendov and P. Marinov, *Verification of the Smale's Mean Value Conjecture for $n \leq 10$* , C. R. Acad. Bulgare Sci. **60** (2007), 1151–1156.
9. M. Shub and S. Smale, *Computational complexity: on the geometry of polynomials and a theory of cost. II*, SIAM J. Comput. **15** (1986), 145–161.
10. S. Smale, *The fundamental theorem of algebra and complexity theory*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), 1–36.
11. D. Tischler, *Critical points and values of complex polynomials*, J. Complexity **5** (1989), 438–456.
12. J. Tyson, *Counterexamples to Tischler's strong form of Smale's mean value conjecture*, Bull. London Math. Soc. **37** (2005), 95–100.

INSTITUTE OF APPLIED MATHEMATICS, FAR-EASTERN BRANCH, RUSSIAN ACADEMY OF SCIENCES,
7 RADIO STREET, VLADIVOSTOK, 690041, RUSSIA
E-mail address: dubinin@iam.dvo.ru

GRADUATE SCHOOL OF INFORMATION SCIENCES, TOHOKU UNIVERSITY, AOBA-KU, SENDAI 980-
8579, JAPAN
E-mail address: sugawa@math.is.tohoku.ac.jp